Volume 15
Number 180 *Disproportionate subclass numbers in tables of multiple classification*

Article 1

March 1935

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March, 1935 Research Bulletin No. 180

Disproportionate Subclass Numbers in Tables of Multiple Classification

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CONTENTS

Disproportionate Subclass Numbers in Tables of Multiple Classification[1]

By George W. Snedecor and Gertrude M. Cox

1. Under the stimulus of some of the newer methods of experimentation there is a decided tendency toward the grouping of classes of data into smaller and more homogeneous subclasses. The weights of swine, for example, may be simultaneously classified according to the sex as well as the litter of the individual animals. Corn yields may be entered in a three-way table by applying the criteria of variety, treatment and soil type. From the resulting tables of multiple classification can be derived information not only of the main effects, such as sex and litter, but also of the interactions between them. Analysis of variance is the most convenient and effective method of reducing such classified data to summary form and testing the significance of the various effects.

The reduction of data is accomplished most easily if the number of items in every subclass is the same. This would be the case if there were taken for analysis five soil samples from each plot of an ordinary block-treatment experiment. There is only a slight increase in the complications of reduction if the subclass numbers are proportional but unequal. This is the situation illustrated in table 4. The proportionality of the numbers of eggs in the pen-shadow groups is observable both in the rows,

$$27:54:36 = 24:48:32, \text{ etc.},$$

and in the columns,

$$27:24:36 = 54:48:72, \text{ etc.}$$

The real difficulties arise when the subclass numbers are disproportionate, as in table 1. The numbers of rats in the sex-generation groups were not under the control of the investigator. Similar situations result from a variety of causes. Animals may be lost by sickness or death. In poultry experiments there is random variation in both the number and sex of birds in each hatch. It is usually impossible to govern the number of schedules returned in a survey of economic or social conditions. The nature of the disadvantages associated with a table such as No. 1 will be fully discussed in the three sections following this one. It is sufficient to note here that in generation 4 the small mean gain may be due entirely to the scarcity of males which are the heavier animals, and may therefore be without biological significance.

The very excellence of the methods for analyzing tables with proportional subclass numbers has served to focus attention

[1]Project No. 346 of the Iowa Agricultural Experiment Station.

TABLE 1. NUMBER AND MEAN GAIN IN WEIGHT (GRAMS)* OF 149 WISTAR RATS DURING 1928-29. FOUR SUCCESSIVE GENERATIONS. GAINS DURING 6 WEEKS BEGINNING AT 28 DAYS OF AGE.

Generation	Male		Female		Total	
	Number of rats	Mean gain	Number of rats	Mean gain	Number of rats	Mean gain
1	21	177	27	110	48	139
2	15	161	25	114	40	132
3	12	156	23	109	35	125
4	7	171	19	107	26	124
Total	55	167	94	110	149	131

*More precise figures are recorded in table 5.

upon the disabilities of tables in which these numbers are disproportionate. It cannot be said that these disabilities have been removed. Only approximate solutions are available. Nevertheless, a gratifying amount of information can be salvaged. It is the purpose of this bulletin to present details of the several available methods of analysis, together with the conditions under which each is valid. Also, a new method will be proposed, that of "expected subclass numbers" (7). This method has been found to give good results under rather varied conditions with only a moderate burden of calculation.

2. At this institution, the effort to analyze the variance of the data in table 1 brought to light some of the peculiarities of disproportionate subclass numbers. These data, furnished by the Foods and Nutrition Section of the Iowa Agricultural Experiment Station, are classified according to the two criteria, sex and generation. Brown (3) and Brandt (1), after a conference with Dr. Fisher, presented tentative analyses of this and similar tables.

There is no difficulty in making the primary analysis of variance, that *within* and *between* the sex-generation groups. The subclass means are efficient estimates of the gains of the animals in these groups. From the original records of individual rat gains table 2 was computed in the usual way (4, section 44) (6, example 2). The mean square *within sex-generation groups* is an appropriate estimate of experimental error.

It is the effort to extend the analysis to the main effects, sex and generation, and to the interactions between them, that leads to difficulties. These are of three kinds. First, the addition theorem for sums of squares does not apply unless the

TABLE 2. PRIMARY ANALYSIS OF VARIANCE OF RAT GAINS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	148	176,836	
Between sex-generation subclasses	7	119,141	17,020
Within subclasses	141	57,695	409

subclass numbers are proportional. This makes it impossible to compute directly the *interaction* sum of squares. Occasionally such a striking situation arises as that presented in table 3. Here, the analysis, partly carried through as usual (6, example 6), leads to sums of squares *between sex* and *between generation* whose sum,

$$114{,}287 + 5{,}756 = 120{,}043,$$

is greater than the *total* sum of squares between the subclasses, 119,141. This is a vivid illustration of the fact that the addition theorem fails, so that the usual way of computing interaction is not available.

3. The second kind of difficulty associated with tables having disproportionate numbers is that of describing the popula-

TABLE 3. ANALYSIS OF VARIANCE OF SEX-GENERATION SUBCLASSES OF RAT GAINS, PARTLY COMPLETED BY THE METHOD APPROPRIATE TO PROPORTIONAL SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between subclasses	7	119,141	17,020
Between sex means	1	114,287	114,287
Between means of generations	3	5,756	1,919
Within subclasses (table 2)	141		409

tion from which the sample may have been drawn. Until this is done, no progress toward analysis is possible. Furthermore, the method of analysis depends upon the character of the assumed population. The sample, therefore, together with the circumstances under which it was collected, must be carefully scrutinized. The following are illustrations of the kinds of hypotheses that may be set up: (i) In the population subclasses the individuals may occur with equal or proportional frequen-

TABLE 4. NUMBER OF EGGS AND MEAN LOSS IN WEIGHT (GRAMS) DURING 6 MONTHS OF STORAGE. PROPORTIONAL SUBCLASS NUMBERS.

Pen	Type of yolk shadow						Total	
	Faint		Medium		Distinct			
	Number	Mean	Number	Mean	Number	Mean	Number	Mean
1	27	2.7	54	2.7	36	2.4	117	2.61
2	24	2.2	48	2.4	32	2.2	104	2.29
3	36	3.0	72	2.4	48	2.7	156	2.63

cies, the disproportionate tabular numbers being due to accidents of sampling. (ii) In the population subclasses interaction may be non-existent.

Some such hypothesis must be at least tentatively formulated before progress with the analysis of variance is possible. The validity of the results depends partly upon the appropriateness of the hypothesis. In the following pages, a good deal of attention will be devoted to the postulates upon which are based the several methods of analyzing variance.

4. A third kind of difficulty introduced by disproportionate subclass numbers is that of estimating the main effects. This appears in two ways. (i) The difference of the sex means, for example, in table 1 is not any simple function of the sex differences in the several generations. In a table like 4, where the subclasses have proportional numbers, the difference between the means of pens 2 and 3, for example, 2.63 — 2.29 = 0.34, is a weighted mean of the differences in the several yolk shadows:—

$$\frac{\frac{(24)\,(36)}{24\,+\,36}\,(3.0\,-\,2.2)\,+\,\frac{(48)\,(72)}{48\,+\,72}\,(2.4\,-\,2.4)\,+\,\frac{(32)\,(48)}{32\,+\,48}\,(2.7\,-\,2.2)}{\frac{(24)\,(36)}{24\,+\,36}\,+\,\frac{(48)\,(72)}{48\,+\,72}\,+\,\frac{(32)\,(48)}{32\,+\,48}} = 0.34,$$

the weights being simple functions of the subclass numbers (2, equation 6). If the attempt is made to apply this formula to the data in such a table as 1, an inequality results:—

$$\frac{\frac{(21)(27)}{21\,+27}\,(177\,-\,110)\,+\,\frac{(15)\,(25)}{15\,+25}\,(161\,-\,114)\,+\,\frac{(12)(23)}{12\,+23}\,(156\,-\,109)\,+\,\frac{(7)(19)}{7\,+19}\,(171\,-\,107)}{\frac{(21)(27)}{21\,+27}\,+\,\frac{(15)(25)}{15\,+25}\,+\,\frac{(12)\,(23)}{12\,+23}\,+\,\frac{(7)(19)}{7\,+19}}$$

is equal to 56.45, which is not exactly equal to the difference between the means, 167 — 110 = 57. While the lack of agreement is not serious in this case, its presence shows that the ordinary theory does not apply, and that the border means may not be good estimates of the main effects. (ii) But no matter what arithmetical method is adopted for computing the main effects, the results are likely to be distorted. For an example, examine table 1 again, comparing the gains in generations 1 and 4. The mean gain of the males in generation 1 is only 6 grams more than in generation 4, while the difference for females is even less. Yet the weighted mean for all rats in generation 4 is 15 grams less than that for generation 1. This is because, in the first generation,

$$\frac{(21)\,(100)}{48} = 44\%$$

of the rats were high gaining males as against only 27 percent in the fourth. Of course if no hypothesis is made concerning the subclass numbers in the population, or if it is assumed that the sample numbers correctly represent those of the population, then the difference of 15 grams between the gains of these two generations must be accepted as the best available estimate. But if the postulate of proportional subclass numbers is set up, as it will be in section 5, this difference is estimated at only 3.9 grams (table 5). Again, when zero interaction in the population is postulated in section 10, the same difference is estimated at 5.4 grams. The reasons for making the postulates will be given later. The reader may then judge as to their validity. At present it is only desired to emphasize the fact that any departure of sample subclass numbers from proportionality (or equality) leads to serious difficulties in estimating the main effects.

Another aspect of this same problem is brought to light by a comparison of the mean square *between generations* in table 3 with the corresponding mean square in tables 6, 8 and 11. In table 3,

$$F^* = 1919/409 = 4.7,$$

indicating highly significant differences among the generations if the sample of table 1 is considered representative. From the later tables, computed under special hypotheses as to the population, the corresponding differences will appear non-significant.

The available methods for analyzing the variance in data with disproportionate subclass numbers will now be explained in some detail. The computations will be presented in such a manner that those untrained in mathematics will be able to follow them. Only such parts of the theory will be introduced as may serve to clarify the discussions in this bulletin. For the remainder, readers will be referred to the original papers.

THE METHOD OF EXPECTED SUBCLASS NUMBERS

5. This method of analyzing tables of multiple classification with disproportionate subclass numbers is based on the assumption that the population from which the sample is drawn really has proportional subclass numbers, the disproportionate numbers in the sample being attributable to the accidents of sampling. The method is available only if every subclass contains at least one observed value.

The method of expected subclass numbers will be illustrated through the medium of the data of table 1. The assumption is that the population numbers of males and females in the several generations were proportional but not necessarily equal. The ratio of the sex numbers seems to have been dictated by the laboratory practice of discarding some of the males while keeping a greater proportion of the females for breeding. The changing total number of rats in the successive generations probably reflects the demands for the animals in experiments other than this. It may be objected that the postulated population has no reality, and might be very different in another series of years—that the real population contains males and females in approximately equal numbers, and that the generations need not be distinguished as to numbers. Granted.

*The use of the statistic, F, as a test of significance is presented in reference (6).

TABLE 5. ACTUAL AND EXPECTED NUMBERS* AND GAINS IN WEIGHT OF 149 RATS.

(Gains are in grams decreased by 100)

Generation		Male		Female		Total	
		Actual	Expected	Actual	Expected	Actual	Expected
1	Number	21	17.7181	27	30.2819	48	48.0000
	Mean	76.952		9.518		39.021	34.41
	Sum		1363.45		288.24	1873	1651.69
2	Number	15	14.7651	25	25.2349	40	40.0000
	Mean	61.467		14.080		31.850	31.57
	Sum		907.56		355.31	1274	1262.87
3	Number	12	12.9195	23	22.0805	35	35.0000
	Mean	55.667		8.522		24.686	25.92
	Sum		719.19		188.16	864	907.35
4	Number	7	9.5973	19	16.4027	26	26.0000
	Mean	71.000		6.790		24.077	30.49
	Sum		681.41		111.37	626	792.77
Total	Number	55	55.0000	94	94.0000	149	149.0000
	Mean	67.327	66.76	9.936	10.03	31.121	30.97
	Sum	3703	3671.61	934	943.08	4637	4614.68

*Each expected number is carried to six significant figures in order to obtain necessary precision in the computations to be explained later. For the same reason the subclass means are calculated more exactly than in table 1. All mean gains are decreased by 100 grams for greater ease in calculation.

Another postulate, that of equal subclass numbers, will be examined in section 15. Afterwards, the consequences of the two assumptions will be discussed. Meanwhile, it is only necessary to keep clearly in mind the hypothesis now set up.

6. The first step is to make a quantitative test of the validity of the assumption of proportional subclass numbers. The usual method of making such a test is by the use of chi-square (5) (4, section 20). The procedure is to compute *expected numbers* in each subclass and compare them with the sample numbers. The expected numbers in any column are calculated by dividing the total for that column into parts proportional to the row totals. For example, those in the first column of table 1 are 55/149 multiplied successively by 48, 40, 35 and 26. The results are displayed in table 5. Next, there is calculated for each subclass the ratio,

$$\frac{(\text{actual number} - \text{expected number})^2}{\text{expected number}},$$

and these ratios are summed for the entire table. In table 5, this sum is

$$\frac{(21 - 17.7181)^2}{17.7181} + \ldots + \frac{(19 - 16.4027)^2}{16.4027} = 2.19,$$

the result being the desired statistic, chi-square. Entering a table of the distribution of chi-square, with degrees of freedom, (number of rows — 1) (number of columns — 1) = (3) (1) = 3, there is found the probability, $P = .54$, that the computed value, 2.19, may be the result of sampling variations in drawing the subclass numbers of table 1 from a population with proportional subclass numbers. Since this is a moderate size for such probability the postulate of proportional numbers is statistically justified. This is the quantitative warrant for proceeding with the method of expected numbers. The computation follows.

7. In each subclass, table 5, the product of the actual mean by the expected number is the *expected sum*. Thus, in the first cell of the table, $(76.952)(17.7181) = 1363.45$. The expected sum and expected mean in the column of totals in generation 1, for example, may now be calculated;

$$1363.45 + 288.24 = 1651.69,$$
$$(1651.69)/48 = 34.41.$$

These expected means in the borders (last row and last column) of the table may be looked upon as better estimates of the main

TABLE 6. ANALYSIS OF VARIANCE OF RAT GAINS. METHOD OF EXPECTED SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between means of subclasses	7	116,305	
Between sex means	1	111,644	111,644
Between means of generations	3	1,480	493
Interactions	3	3,181	1,060
Within subclasses (table 2)	141		409

effects, under the assumption of proportional subclass numbers in the population, than were the original means—they include adjustments to counteract the difficulty discussed in section 4.

The analysis of variance is now carried through with the proportional expected subclass numbers and expected sums in the usual fashion (6, example 6). The results are displayed in table 6.

8. There are now apparent two interesting consequences of the method of expected subclass numbers together with the hypothesis upon which it rests. (i) An estimate of interaction has been made. Since

$$F = 1060/409 = 2.6,$$

the probability that the postulated population is free from interaction is almost 5 percent, so that the existence of interaction is only tentatively established. (ii) The generation mean gains do not differ significantly among themselves. This is contrary to the conclusion that would be drawn from table 3 where no hypothesis concerning population was made. The differing results in tables 3 and 6 are a direct consequence of the postulate of proportional subclass numbers in the population. The reader need not feel obliged to draw final conclusions on this point at present because further evidence is to be presented.

The method of expected subclass numbers has been tried under rather diverse conditions during the last year, and has been found to yield satisfactory results in all cases where the requirements of proportional population numbers has been even approximately fulfilled. Since the experimental evidence supporting the validity of this method is best presented in comparison with the results obtained by other methods, it will be reserved (section 32) until these other methods have been discussed.

9. Before leaving this topic, it should be noted that any two classes of the main effects may be compared by the use of their expected means. This is owing to the fact that, under the hypothesis of proportional numbers in the parent population, the expected means are the best available estimates of these main effects. As an example, the difference between the mean gains of generations 1 and 4 is estimated as

$$34.41 - 30.49 = 3.92 \text{ grams},$$

a result which was anticipated in section 4.

THE METHOD OF FITTING CONSTANTS

10. This is a method in which the theory of least squares is applied to the solution of tables of double classification with disproportionate subclass numbers. The method was devised by Fisher (4) for one of his students before 1931, but no published account of the results has been found. A verbal account of the method, communicated to Brandt in 1931, was published by him in 1933 (2). Meanwhile, Yates (8) had perfected the technique, and extended its application to include all two-way tables without restriction as to the number of classes. This is the only method available in cases where some of the subclasses are devoid of data.

The fundamental assumption in the method of fitting constants is that there is no interaction in the population from which the sample is drawn. Under this postulate a set of constants is fitted to the data with these conditions; (i) The constants determine a set of subclass means with zero interaction (6, section 45). (ii) The sum of the squares of the differences between these and the actual means is a minimum. The theory underlying this method has been well presented by Yates (9).

An illustration of the nature of the fitted constants will be found in section 13. It is noteworthy that, despite the postulate of zero interaction, this method affords a test of significance for interaction if present.

Two examples will be worked out. In the first, a relatively short one, the method will be applied to the data of table 1 despite the fact that interaction may exist in the postulated population. In the second (appendix II), application will be made to a sample in which the hypothesis of zero interaction is justified, but in which the hypothesis of proportional subclass numbers is untenable.

11. In table 7 certain information is copied from table 5. The constants to be fitted are as follows.

1. A general mean, m.
2. Two sex constants, a_1 and a_2, where $a_1 + a_2 = 0$.
3. Four generation constants, b_1, b_2, b_3, b_4, where $b_1 + b_2 + b_3 + b_4 = 0$.

The nature of these constants will be exhibited in table 9, after the computation is finished.

TABLE 7. INFORMATION CONCERNING GAINS IN WEIGHT OF 149 RATS, TRANSCRIBED FROM TABLE 5. METHOD OF FITTING CONSTANTS

Generation constants	Male, a_1		Female, a_2		Total			
	Number	Ratio	Number	Ratio	Number	Sum of gains	Mean gain	b + m
b_1	21	0.437500	27	0.562500	48	1873	39.021	42.569
b_2	15	0.375000	25	0.625000	40	1274	31.850	38.947
b_3	12	0.342857	23	0.657143	35	864	24.686	33.607
b_4	7	0.269231	19	0.730769	26	626	24.077	37.179
Total	55		94		149			
Sum of gains	3703		934			4637		
Sex constants		28.3873		−28.3873				

Each ratio in the table is the subclass number divided by the generation number; that is, 21/48 = 0.437500, 27/48 = 0.562500, 15/40 = 0.375000, etc. A check on calculation is provided by the fact that in each line the sum of these ratios is 1.000000.

The coefficients of two *a-equations* are calculated as follows:

$$[(21)(0.437500) + (15)(0.375000) + (12)(0.342857) + (7)(0.269231) - 55]a_1$$
$$+ [(21)(0.562500) + (15)(0.625000) + (12)(0.657143) + (7)(0.730769)]a_2$$
$$= (21)(39.021) + (15)(31.850) + (12)(24.686) + (7)(24.077) - 3703.$$
$$[(27)(0.437500) + (25)(0.37500) + (23)(0.342857) + (19)(0.269231)]a_1$$
$$+ [(27)(0.562500) + (25)(0.625000) + (23)(0.657143) + (19)(0.730769) - 94]a_2$$
$$= (27)(39.021) + (25)(31.850) + (23)(24.686) + (19)(24.077) - 934.$$

These two equations reduce to the same one—

$$-34.1886\, a_1 + 34.1886\, a_2 = -1941.05.$$

It isn't really necessary to compute both equations, but one serves as a check on the other. Since $a_1 + a_2 = 0$, then $a_2 = -a_1$. Substituting in the above equation and solving, there results,

$$a_1 = 28.3873,$$
$$a_2 = -28.3873.$$

For convenience, these values are entered in the last line of table 7.

The next set of equations involves both m and the generation constants.

$$b_1 + m = 39.021 - (0.437500)(28.3873) + (0.562500)(-28.3873) = 42.569,$$
$$b_2 + m = 31.850 - (0.375000)(28.3873) + (0.625000)(-28.3873) = 38.947,$$
$$b_3 + m = 24.686 - (0.342857)(28.3873) + (0.657143)(-28.3873) = 33.607,$$
$$b_4 + m = 24.077 - (0.269231)(28.3873) + (0.730769)(-28.3873) = 37.179.$$

These quantities, $b + m$, are now entered in the right hand column of table 7.

Although the analysis of variance can now be completed, still it is interesting to solve specifically for the generation constants. Adding the four equations, remembering that $b_1 + b_2 + b_3 + b_4 = 0$, the result is,

$$4m = 152.302,$$
$$m = 38.0755.$$

The values of the generation constants are calculated by substituting this value of m in the four equations above, whence,

$$b_1 = 4.4935,\ b_2 = 0.8715,\ b_3 = -4.4685,\ b_4 = -0.8965,$$

the sum of these values being zero. These figures will be used

TABLE 8. ANALYSIS OF VARIANCE OF RAT GAINS. METHOD OF FITTING CONSTANTS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between sex means	1	110,202	110,202
Between means of generations	3	1,671	557
Interactions	3	3,183	1,061
Within subclasses (table 2)	141		409

later. Meanwhile, it is easier to revert to the values of $b + m$ to complete the computation.

12. The next calculation is that of a constant analogous to $R^2\Sigma(x - \bar{x})^2$ in multiple regression—the "reduction in sum of squares due to fitting constants." The amount is,

$$(28.3873)(3703) + (-28.3873)(934) + (42.569)(1873) + \ldots + (37.179)(626) - (4637)^2/149 = 115{,}958,$$

the subtracted amount, $(4637)^2/149$, being the usual correction for mean.

The final steps in the computation require the results of the partly computed analysis of table 3. From that table,

Sum of squares between sex,	114,287
Sum of squares between generation	5,756
Sum	120,043
Reduction due to constants,	115,958
Correction for disproportionate subclass numbers	4,085

This correction subtracted from each of the two sums of squares, sex and generation, results in the values recorded in table 8. For example, for sex, 114,287 — 4,085 = 110,202. The sum of squares for the interactions is,

Sum of squares between subclasses (table 3)	119,141
Reduction due to fitting constants,	115,958
Sum of squares for interactions,	3,183

The analysis of variance in table 8 is now completed in the ordinary way. The conclusions to be drawn are identical with those from table 6.

13. In fitting constants, the nature of the assumptions is observable in table 9. The computed mean in each subclass is the sum of the general mean, m (to which must be added the 100 grams deducted for ease in computation), and the two constants in the same row and same column. For example, the computed mean for males in generation 3 is,

$$138.08 + 28.39 - 4.47 = 162 \text{ grams.}$$

Again, for females in generation 2,

$$138.08 - 28.39 + 0.87 = 110 \text{ grams, approximately.}$$

A slight correction was necessary in rounding decimals in the last figure in order to balance the table. It is clear that there is zero interaction among the computed means (section 30), since,

$$171 - 167 = 114 - 110, \text{ etc.}$$

Also, it is apparent that the computed means vary less among themselves than do the actual means. The variation of the latter gives rise to the sum of squares between subclasses, while that of the former leads to the sum of squares due to fitting constants. The difference is therefore attributed to the presence of interactions.

14. Under the hypothesis of zero interaction in the population, the estimate of the sex difference is

$$28.39 - (-28.39) = 56.78 \text{ grams.}$$

Similarly, the estimate of the difference between generations 1 and 4, for example, is

$$4.49 - (-0.90) = 5.39 \text{ grams,}$$

a result which was referred to in the discussion of section 4.

TABLE 9. MEAN GAINS OF RATS (GRAMS) COMPUTED FROM FITTED CONSTANTS.

(Actual means in parentheses.)

General mean = 100 + 38.08 = 138.08 grams

Generation	Generation constants	Sex and sex constants	
		Male, 28.39	Female, −28.39
1	4.49	171 (177)	114 (110)
2	0.87	167 (161)	110 (114)
3	−4.47	162 (156)	105 (109)
4	−0.90	166 (171)	109 (107)

THE METHOD OF WEIGHTED SQUARES OF MEANS

15. This method was devised by Yates (8, 9) to analyze samples under the hypothesis of existent interactions in the parent population. It furnishes estimates of the main effects in tables of double classification. In the special case of a table with only two classes one way, it also furnishes a test of the significance of the interactions,[3] the test being identical with that offered by the method of fitting constants. One of the fundamental assumptions in Yates' theory is that the unweighted means of the subclasses furnish valid estimates of the class means. It would seem, therefore, that the method is especially appropriate if the postulated population has equal subclass numbers. This conclusion is supported by the following argument: if the method is applied to a sample with equal subclass numbers it yields exactly the same results as the standard method for such numbers; but if it is applied to a sample with proportional (but not equal) subclass numbers the results do not coincide with those obtained from the standard method for proportional numbers. This matter will be discussed further in connection with the experimental evidence in section 31.

For the sake of comparison, the computation of the method of weighted squares of means also will be illustrated with the data of table 1. The appropriateness of the assumption of equal subclass numbers in the population was discussed in section 5. The numbers and means in table 10 were transcribed from table 5.

16. First, the reciprocal of the subclass number is entered in each cell of the table. Then these reciprocals are summed in the rows and in the columns.

Next, the unweighted means are computed in rows and columns. As examples,

$$(76.952 + 9.518)/2 = 43.235,$$

$$(76.952 + 61.467 + 55.667 + 71.000)/4 = 66.271.$$

Also, the unweighted differences are entered at the right;

$$76.952 - 9.518 = 67.434, \text{ etc.}$$

In the column and row designated by "weight" are entered the reciprocals of the sums of reciprocals; that is,

$$1/0.08466 = 11.8120,$$

$$1/0.34048 = 2.9370.$$

The weights in the column are those of the corresponding differences, and one fourth of the weights of means; the weights of the means being the products of the tabular weights by the

[3]Mr. Walter A. Hendricks, as a minor feature of a paper recently accepted for publication by "The Annals of Mathematical Statistics," has suggested a method for obtaining a test of the significance of the interaction in any two-way table when the method of weighted squares of means is used. The paper, entitled "Analysis of Variance Considered as an Application of Simple Error Theory," was expected to appear in the issue of December, 1934. The method furnishes two tests of the interaction in a table. While the two tests do not, in general, agree, they are not likely to differ materially. One test of interaction is obtained by computing independently the two quantities, (i) sum of squares between means of rows, and (ii) sum of squares between means of rows within columns, then testing the significance of the ratio of the latter to the former. (See reference 6, example 7.) The first sum of squares is computed by the method of weighted squares of means, the second by the usual method for unequal numbers of observations in the classes (6, example 2).

square of the number of sex classes. The weights of the sex means, if desired, are obtained by multiplying the tabular weights in the last row but one by 16, the square of the number of generations.

TABLE 10. INFORMATION ABOUT GAINS OF RATS TRANSCRIBED FROM TABLE 5. METHOD OF WEIGHTED SQUARES OF MEANS.

Generation		Sex		Unweighted mean and sum of reciprocals	Difference between means	Weight	Products	
		Male	Female				Mean	Difference
	Subclass number	21	27					
1	Mean	76.952	9.518	43.235	67.434			
	Reciprocal	0.04762	0.03704	0.08466		11.8120	510.697	796.529
	Subclass number	15	25					
2	Mean	61.467	14.080	37.773	47.387			
	Reciprocal	0.06667	0.04000	0.10667		9.3747	354.114	444.236
	Subclass number	12	23					
3	Mean	55.667	8.522	32.094	47.145			
	Reciprocal	0.08333	0.04348	0.12681		7.8858	253.089	371.776
	Subclass number	7	19					
4	Mean	71.000	6.790	38.895	64.210			
	Reciprocal	0.14286	0.05263	0.19549		5.1154	198.962	328.463
Total	Unweighted mean	66.271	9.727					
	Sum of reciprocals	0.34048	0.17315	Sum of weights = 34.1879			1316.862	1941.004
	Weight	2.9370	5.7753	Sum of weights = 8.7123				
	Product	194.639	56.190	Sum of products = 250.829				

In the last two columns are the products of the weights by the corresponding means and differences. As examples,

$$(43.235)\ (11.8120) = 510.697,$$
$$(67.434)\ (11.8120) = 796.529.$$

Similarly, in the last row are such products as,

$$(66.271)\ (2.9370) = 194.639.$$

All the above quantities, except means and differences, are summed in the rows and columns. The total of the reciprocals is not entered in the table, but the sum in the column of reciprocals must check with that in the corresponding row.

From the above totals are now computed the following weighted means:—

$$\text{Of sexes: } 250.829/8.7123 = 28.7902.$$
$$\text{Of generations: } 1316.862/34.1879 = 38.5184.$$
$$\text{Of differences: } 1941.004/34.1879 = 56.7746.$$

Finally, the *weighted squares of means* are:—

$$\text{Sex: } 16[(66.271)\ (194.639) + (9.727)\ (56.190) - (28.790)\ (250.829)] = 99596.$$

$$\text{Generation: } 4[(43.235)(510.697 + \ldots + (38.895)(198.962) - (38.518)\ (1316.862)] = 2377.$$

$$\text{Differences: } (67.434)\ (796.529) + \ldots + (64.210)\ (328.463) - (56.775)\ (1941.004) = 3182.$$

The numbers, 16 and 4, are the squares of the numbers of classes in each sex or generation total. They convert the weights of differences into those of the corresponding means. The terms subtracted are the usual correction terms, necessary to convert the sums of squares of observations into sums of squares of deviations from mean.

The weighted squares of means are entered in table 11 as sums of squares. The analysis of variance is completed in the usual manner. (The mean square for *interactions* is not an estimate, but is used for testing significance.)

The conclusions to be drawn from table 11 are identical with those from table 6. While the estimate of the mean square

TABLE 11. ANALYSIS OF VARIANCE OF RAT GAINS. METHOD OF WEIGHTED SQUARES OF MEANS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between means of sexes	1	99,596	99,596
Between means of generations	3	2,377	792
Interactions	3	3,182	1,061
Within subclasses (table 2)	141		409

between generations is larger by the method of weighted squares of means, it is not large enough to be significant of population differences.

17. It is not usually proper to compare class means after the demonstration of homogeneity in the population. For the sake of the illustration, however, the difference between the unweighted means of the first and second generations may be tested. This difference is

$$43.235 - 37.773 = 5.462 \text{ grams.}$$

The standard error of this mean difference is derived from experimental error, as usual, together with the weights of the two means. These latter are four times the weights of the differences in table 10; that is, 47.2480 and 37.4988, respectively. Whence, the required standard error is

$$\sqrt{(409)\,(1/47.2480 + 1/37.4988)} = 4.423.$$

The resulting value of t is,

$$t = 5.462/4.423 = 1.235.$$

based on 141 degrees of freedom.

THE METHOD OF UNWEIGHTED MEANS

18. Yates (8, 9), who suggested this method, described it as a "simple approximate method". He indicated that it can be relied upon only if the subclass numbers are approximately equal, and presumably represent a population with equal numbers. The computation being relatively easy, the method is

TABLE 12. MONTHLY MEAN GAINS IN WEIGHT (POUNDS) OF HOLSTEIN HEIFERS AT THREE AGES.

Month of gain	Age in months						Total Number
	13		18		24		
	Number	Gain	Number	Gain	Number	Gain	
August	5	22.2	7	21.4	3	40.7	15
September	8	24.4	3	30.3	1	39.0	12
October	4	45.0	3	30.3	3	48.3	10
November	4	18.0	3	67.0	6	49.0	13

readily available for the survey of samples when no tenable hypothesis about the population presents itself.

The data in table 12 were abstracted from some more extensive information tabulated by Dwight Espe of the Iowa Agricultural Experiment Station. The object of the preliminary analysis is to find out if significant interactions exist, and if the monthly gains differ significantly with the advancing season.

The analysis of the variance of the *means* in table 12, no attention being paid to the subclass numbers, is carried on in the usual manner (6, example 3). The results are entered in table 13. The only novelty is the method of estimating experimental error. This is done in three steps. First, the mean square gain *within subclasses* is computed in the regular way (6, example 2) from the individual monthly gains of the heifers. The result appears in table 14. Second, an average subclass number, N, is calculated by using the harmonic mean of the 12 subclass numbers in table 12. This is given by,

$$\frac{1}{N} = \frac{1}{12}\left(\frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3} + \frac{1}{6}\right) = 0.3168.$$

Finally, the variance of the means in table 12 is computed by dividing the variance of the individual gains by their average number; that is,
(Mean square *within subclasses*)/N = (931.8) (0.3168 = 295.2. This is the appropriate experimental error to be used with the mean squares of table 13.

19. Since no significant differences are indicated by this analysis, there is no occasion for further investigation. Com-

TABLE 13. ANALYSIS OF VARIANCE OF THE MEAN GAINS OF HEIFERS. METHOD OF UNWEIGHTED SUBCLASS MEANS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	11	2,319	
Between means of ages	2	573	287
Between means of months	3	561	187
Interactions	6	1,185	198
Experimental error	38		295

TABLE 14. PRIMARY ANALYSIS OF VARIANCE OF INDIVIDUAL GAINS OF 50 HEIFERS IN 12 SUBCLASSES.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between subclasses	11	9,522	
Within subclasses	38	35,409	931.8
Total	49	44,931	

pare reference (6, section 72). If the results of this analysis had raised some critical questions, it would be necessary to continue the examination by the use of that one of the more adequate methods which seemed to be most appropriate.

DISCUSSION OF METHODS

20. In the matter of ease of computation, there is really less choice among the methods than has, perhaps, been apparent. The primary analysis is required in all cases, and this may be more time-consuming than the specialized remainder. If they are arranged in order of ease of computation, starting with the easiest, the methods occupy this sequence; unweighted means, expected subclass numbers or weighted squares of means, and fitting constants.

21. The least squares method of fitting constants requires the fewest assumptions, is most general in its range of applications, and should, therefore, be looked upon as the standard of comparison. It is supposed to furnish only a test of the significance of the interactions, not an estimate. This theoretical fact, however, seems to be unimportant in practice. The evidence which has been presented, taken in connection with that to come, strongly supports the conviction that the mean square for testing the interactions is also an excellent practical estimate of their magnitude. From the fundamental assumption of non-existent interactions in the population, it might be supposed that estimates of the main effects from the sample would lose validity as the size of the interactions increased. This, however, does not seem to be the case. Fitting constants is the only method applicable if data are lacking in some of the subclasses. On the other hand, this method requires a rather unusual type of computation. If the number of classifications is great, the burden of calculation becomes forbidding. And finally, for tables where there are more than two criteria of classification, the method of fitting constants has not been completely developed.

22. The method of expected subclass numbers is presented as a convenient and reliable substitute for fitting constants, even more flexible in its range of applications than is indicated by its hypothesis of proportional numbers in the parent popula-

tion. With the exception of a small amount of preliminary calculation, it follows exactly the usual scheme of computation familiar to all users of analysis of variance. Under its fundamental hypothesis it affords an estimate of both the main effects and the interactions. It is applicable to cases in which there are more than two criteria of classification (appendix I). In only one example has the postulate of proportional numbers in the population been found untenable. In that case, chi-square had the value of 348 in a 5×9 table (appendix II).

23. The method of weighted squares of means has the authority of an excellent theoretical background, and involves no great amount of computation. Its postulate of equal subclass numbers in the several classes of the population is easily set up in a great variety of problems. Even when the postulate falls, the results of the analysis are usually reliable. The method does not, however, yield either a test or an estimate of interaction except in tables with only two classes in one direction. Its estimates of the main effects seem to lose validity somewhat as the subclass numbers depart from proportionality (section 32). If the subclass numbers are proportional but unequal, the method fails to produce the results yielded by the standard method of analysis of variance (section 31).

24. The method of unweighted means is intended as an approximation, easily computed. It can be relied upon if the subclass numbers depart only slightly from equality. It may be used sometimes, especially in a preliminary survey, if none of the other methods seem to apply. On its results may be based the hypothesis of interactions present or absent; then, if any critical questions have arisen, the more suitable methods of analysis may be applied. In many cases, however, the results are so conclusive that further investigation is unnecessary.

25. In section 5, a question was raised concerning the appropriateness of an assumption made about a population. This is an insistent question, because the validity of the results depends, theoretically at least, upon the fitness of the hypothesis. Fortunately, it appears that in many cases one assumption is about as good as another. In treating the data of table 1, three methods yielded results whose interpretations are identical. In ordinary practice, the investigator may be confident that any reasonable postulate about the population will be adequate for his needs. There are only a few cases in which more than one method need be employed before conclusions can be reached.

But, after all, the appropriateness of any of the postulates underlying the special methods may be called in question. If the sample is large, and if its subclass numbers and means are believed to represent correctly the facts in the population, then no special assumptions can be made. In such cases the primary analysis of variance (table 2) is valid, as are the separate analyses of the main effects (table 3). The presence or absence of interaction may be tested by fitting constants.

26. In the case of the rat data of table 1, the rejection of all special hypotheses would lead to the conclusion that the gains of the rats differ significantly in the several generations, rather a disconcerting deduction for the investigator in nutrition. While there is nothing in the data themselves to require special assumptions, yet from the laboratory viewpoint, the postulate of equal sex-generation numbers seems most reasonable. There is every reason to believe that the sex ratio varies only as in random sampling, and that the numbers of individuals retained in the successive generations were largely matters of laboratory requirements.

27. The question of a valid hypothesis of population is related to that of the representativeness of a sample. From the standpoint taken in this bulletin the question takes this form, "Of what population is this sample to be considered representative?" Not infrequently the question may be answered in more than one way. As an example may be cited a sample of swine slaughtered at Iowa State College. In this sample males predominated roughly in the ratio of three to one, the females having been retained on the farm for breeding. If the aggregate of locally slaughtered swine is the population postulated, then the proportional class numbers are representative. But if the swine population of the locality is being investigated, a postulate of equal class numbers in the sexes is more appropriate.

EXPERIMENTAL PROCEDURE AND RESULTS

28. The object of the experimental part of this project was to learn how the method of expected subclass numbers would compare with other methods for analysing samples with disproportionate subclass numbers drawn from a population of known constitution. The individuals of the population were assumed to be classified according to two independent criteria each with three classes, such as may be described by a 3×3 table. The variate was normally distributed in each of the nine subclasses. The subclass numbers were proportional.

To represent this postulated population, values of the normal deviate were taken at random from table 1 in Pearson's "Tables for Statisticians and Biometricians."[4] Tippett's "Random Sampling Numbers"[5] were used to insure random sampling. The deviates were distributed among the nine subclasses till the proportional numbers indicated in table 15 were reached. To each value in the first subclass was added such a constant as would make the mean of that group approximately equal to 9. The means of the other subclasses were similarly fixed. From this large sample were drawn the small samples, to be des-

[4]Issued by the Biometric Laboratory, University College, London.
[5]Tracts for Computers, No. XV, Cambridge University Press, London.

TABLE 15. LARGE SAMPLE CONSTITUTING THE POPULATION FROM WHICH SMALL SAMPLES WERE DRAWN. SERIES I.

Row		Column			Total
		A_1	A_2	A_3	
B_1	Number Mean	200 8.9994	400 8.9863	280 8.9931	880
B_2	Number Mean	400 8.0060	800 7.0015	560 7.9877	1,760
B_3	Number Mean	300 7.9885	600 7.9959	420 6.9901	1,320
Total	Number	900	1,800	1,260	3,960

cribed below, with disproportionate subclass numbers. The analysis of variance of the large sample appears in table 16.

As a first step towards drawing the final small samples, the subclasses in the population were assumed to be divided into 20 equal parts. The subclass numbers of one of these parts are indicated in table 17. This part is designated as a *theoretical small sample.* The analysis of its variance is recorded in tables 19-21. Next, a set of disproportionate subclass numbers was determined in this way: a maximum departure of 10 percent (for example) was allowed from each subclass number in the theoretical small sample. This possible departure is indicated in each cell of table 17. Then the actual departure from proportionality was taken at random from among the possible departures in each subclass, Tippett's "Random Sampling Numbers" being used again. A set of departures resulting from one such procedure is recorded in the table. As a final step, values of the normal variate were placed at random in the several subclasses in the numbers just determined. The results of a drawing with 50 percent possible departures are exhibited in table 18.

29. From the description given it will be understood that the final sample with disproportionate subclass numbers is drawn from the population with two random features; (i) the individual values of the normal variate are taken at random, and (ii) the departure of the subclass numbers from proportionality is random. It is believed that both these features enter into most samples having disproportionate subclass numbers.

TABLE 16. ANALYSIS OF VARIANCE OF LARGE SAMPLE CONSTITUTING POPULATION FROM WHICH SMALL SAMPLES WERE DRAWN. SERIES I

Source of variation	Degrees of freedom	Sum of squares	Mean square
Within subclasses	3,951	4,112.7	1.04
Between means of rows	2	1,338.9	669.5
Between means of columns	2	121.5	60.8
Interactions	4	597.9	149.5
Total	3,959	6,171.1	

TABLE 17. PROPORTIONAL NUMBERS IN THE THEORETICAL SMALL SAMPLES AND STEPS IN DRAWING DISPROPORTIONATE NUMBERS IN A SAMPLE WITH 10 PERCENT POSSIBLE DEPARTURES.

Row		Column A_1	A_2	A_3
B_1	Theoretical	10	20	14
	Possible departure	±1	±2	±1
	Actual departure	−1	+1	−1
	Subclass number	9	21	13
B_2	Theoretical	20	40	28
	Possible departure	±2	±4	±3
	Actual departure	−1	+4	0
	Subclass number	19	44	28
B_3	Theoretical	15	30	21
	Possible departure	±2	±3	±2
	Actual departure	−2	−3	−2
	Subclass number	13	27	19

It seems reasonable to think that many sampled populations have proportional numbers in their subclasses, and that disproportionate numbers are accidents of sampling. Such is the reasoning that led to the chosen method of drawing the experimental samples.

Each of the 28 samples drawn was analyzed by three methods, expected numbers, fitting constants, and Yates' method of weighted squares of means. The results, designated as series I, appear in tables 19 to 21.

After the samples of series I had been completed, the mean of the A_2 B_2 subclass was changed to 8.0015 and 16 more samples drawn. The object was to decrease the interaction. The analysis of the theoretical small sample from the second population appears in tables 22 and 23. The samples of this series II were not analyzed by the method of weighted squares of means.

TABLE 18. SMALL SAMPLE WITH 50 PERCENT POSSIBLE DEPARTURE FROM PROPORTIONAL SUBCLASS NUMBERS.
SAMPLE NO. 4, SERIES I.

Row		Column A_1	A_2	A_3	Total
B_1	Number	8	24	12	44
	Mean	8.17	9.04	9.16	8.92
B_2	Number	14	30	33	77
	Mean	7.62	7.19	7.92	7.58
B_3	Number	17	21	23	61
	Mean	7.72	8.09	7.07	7.60
Total	Number	39	75	68	182
	Mean	7.78	8.04	7.85	7.91

In addition to the more formal experimentation already described, a number of minor analyses were made in order to test and illustrate particular assumptions. Two of the more interesting of these will be presented.

DISPROPORTIONATE SUBCLASS NUMBERS IN A TABLE WITH ZERO INTERACTION

30. Table 24 exhibits a purely arbitrary sample in which the means of the subclasses are chosen so that the interactions are zero; that is,

$$2 - 3 = 3 - 4 = 7 - 8,$$
$$3 - 7 = 4 - 8 = 8 - 12, \text{ etc.}$$

The disproportionate subclass numbers closely approximate proportionality. The object is to determine how well the zero interaction can be estimated by the various methods of analysis, and to compare the estimates of the main effects.

The results appear in table 25. They show that in this case the zero interaction of the sample is correctly estimated by the two methods, expected numbers and fitting constants. There is, of course, no estimate of interaction by the method of weighted squares of means. As for the main effects, the methods of expected numbers and fitting constants yield almost identical

TABLE 19. ANALYSES OF SAMPLES DRAWN AT RANDOM FROM A POPULATION DESCRIBED BY A 3×3 TABLE. SERIES I. TEN PERCENT POSSIBLE DEPARTURE FROM THEORETICAL SUBCLASS NUMBERS.

Sample	Chi-square	Source of variation	Mean square			
			Original	Expected numbers	Fitting constants	Weighted squares
Theoretical	9.5 at 5% point	Within	1.0			
		Rows	33.5			
		Columns	3.0			
		Interaction	7.5			
1	0.3	Rows	38.7	39.1	39.6	33.3
		Columns	5.1	5.8	6.0	5.3
		Interaction		6.9	6.8	
2	0.2	Rows	19.7	19.8	19.9	16.4
		Columns	12.4	12.6	12.6	10.9
		Interaction		11.5	11.2	
3	0.2	Rows	44.8	45.6	45.7	38.6
		Columns	6.7	7.6	7.6	6.0
		Interaction		7.4	7.3	
4	0.3	Rows	34.8	34.4	35.4	23.5
		Columns	1.6	2.1	2.2	0.2
		Interaction		9.4	9.2	
5	0.1	Rows	46.2	45.7	43.2	39.8
		Columns	6.5	3.6	3.6	1.8
		Interaction		7.6	7.6	
6	0.1	Rows	30.5	30.4	30.5	24.0
		Columns	1.3	1.2	1.3	0.8
		Interaction		5.7	5.7	
7	0.2	Rows	64.2	62.7	64.4	51.2
		Columns	0.7	0.8	0.9	0.1
		Interaction		15.5	15.5	
8	0.2	Rows	34.7	34.8	34.7	30.9
		Columns	2.5	2.3	2.5	2.8
		Interaction		11.1	11.1	
Average		Rows	39.2	39.1	39.2	32.2
		Columns	4.6	4.5	4.6	3.5
		Interaction		9.4	9.3	

results, while those obtained by the method of weighted squares of means are considerably smaller. In fact, if this were a random sample from a normal population, the mean squares for rows and columns would be adjudged significantly greater than that within subclasses if either of the first two methods are used, but not significantly greater by the method of weighted squares of means.

TABLE 20. ANALYSES OF SAMPLES DRAWN AT RANDOM FROM A POPULATION DESCRIBED BY A 3×3 TABLE. SERIES I. TWENTY-FIVE PERCENT POSSIBLE DEPARTURE FROM THEORETICAL SUBCLASS NUMBERS.

Sample	Chi-square	Source of variation	Mean square			
			Original	Expected numbers	Fitting constants	Weighted squares
Theoretical	9.5 at 5% point	Within	1.0			
		Rows	33.5			
		Columns	3.0			
		Interaction	7.5			
1	1.6	Rows	33.5	35.7	34.6	30.2
		Columns	2.6	4.4	3.6	4.7
		Interaction		9.3	8.9	
2	3.0	Rows	46.3	45.2	46.1	37.2
		Columns	0.5	0.4	0.3	0.2
		Interaction		8.5	8.7	
3	1.4	Rows	57.5	55.9	54.7	50.1
		Columns	8.1	5.5	5.2	4.5
		Interaction		7.5	7.7	
4	3.3	Rows	22.3	20.9	20.1	20.2
		Columns	7.8	6.0	5.6	3.0
		Interaction		8.9	9.4	
5	1.3	Rows	24.3	24.9	24.2	22.1
		Columns	9.4	9.0	9.2	9.3
		Interaction		8.3	8.2	
6	0.6	Rows	34.3	33.9	35.0	25.7
		Columns	1.9	2.2	2.6	1.1
		Interaction		11.4	11.1	
7	3.8	Rows	15.3	17.5	15.1	14.2
		Columns	1.5	1.6	1.3	0.8
		Interaction		9.1	9.2	
8	2.5	Rows	34.0	34.7	35.1	26.3
		Columns	2.7	3.7	3.8	3.7
		Interaction		10.1	10.0	
9	2.0	Rows	53.6	53.5	54.7	49.6
		Columns	3.8	4.4	4.9	3.4
		Interaction		7.6	7.5	
10	2.0	Rows	25.0	24.6	23.4	23.0
		Columns	9.8	8.7	8.2	6.7
		Interaction		5.1	5.0	
11	3.0	Rows	49.2	47.8	47.4	41.2
		Columns	5.1	4.0	3.3	4.0
		Interaction		9.2	9.2	
12	0.1	Rows	44.4	44.0	44.7	31.1
		Columns	2.4	2.7	2.7	1.1
		Interaction		9.5	9.4	
Average		Rows	36.6	36.6	36.2	30.9
		Columns	4.6	4.4	4.2	3.5
		Interaction		8.7	8.7	

PROPORTIONAL SUBCLASS NUMBERS IN A SAMPLE WITH INTERACTION EXISTING

31. The sample of table 26 is an illustration of proportional subclass numbers with interactions present. Since the standard method of analysis of variance applies, there is no question of the efficiency of the estimates in the first column of table 27. The method of expected numbers is identical with the standard method. The object of applying the other methods is to determine how successfully the known results can be estimated by their use. A table was chosen with only two classes in one direction in order that a mean square for interactions might be obtained by the method of weighted squares of means. The results indicate the known facts that when sublcass numbers are proportional, (i) the method of fitting constants yields an analysis identical with that of the regular method, (ii) the method of weighted squares of means produces the same estimate of interaction as the method of fitting constants, and (iii)

TABLE 21. ANALYSES OF SAMPLES DRAWN AT RANDOM FROM A POPULATION DESCRIBED BY A 3×3 TABLE. SERIES I. FIFTY PERCENT POSSIBLE DEPARTURE FROM THEORETICAL SUBCLASS NUMBERS.

Sample	Chi-square	Source of variation	Mean square			
			Original	Expected numbers	Fitting constants	Weighted squares
Theoretical	9.5 at 5% point	Within	1.0			
		Rows	33.5			
		Columns	3.0			
		Interaction	7.5			
1	9.5	Rows	28.4	27.2	24.7	25.2
		Columns	6.8	3.1	3.1	0.8
		Interaction		13.0	12.8	
2	4.5	Rows	25.5	27.6	26.2	27.0
		Columns	4.4	5.1	5.0	4.4
		Interaction		4.1	4.0	
3	15.1	Rows	34.0	28.8	28.6	27.7
		Columns	11.5	6.3	6.1	3.5
		Interaction		9.1	9.2	
4	6.3	Rows	29.4	28.9	28.5	20.0
		Columns	1.1	0.4	0.2	0.9
		Interaction		6.6	6.3	
5	2.1	Rows	30.8	31.5	28.8	22.7
		Columns	7.2	6.2	5.2	1.9
		Interaction		6.3	6.5	
6	8.5	Rows	18.7	21.2	18.8	20.0
		Columns	2.5	3.8	2.6	2.8
		Interaction		4.2	3.9	
7	10.0	Rows	37.6	40.2	36.0	31.2
		Columns	4.7	4.2	3.2	2.4
		Interaction		7.0	7.0	
8	5.8	Rows	19.5	19.7	18.6	15.6
		Columns	4.6	4.8	3.6	4.0
		Interaction		8.7	8.4	
Average		Rows	28.0	28.1	26.3	23.7
		Columns	5.4	4.2	3.6	2.6
		Interaction		7.4	7.3	

TABLE 22. ANALYSES OF SAMPLES DRAWN AT RANDOM FROM A POPULATION DESCRIBED BY A 3×3 TABLE. SERIES II. TEN PERCENT POSSIBLE DEPARTURE FROM THEORETICAL SUBCLASS NUMBERS.

Sample	Chi-square	Source of variation	Mean square		
			Original	Expected numbers	Fitting constants
Theoretical	9.5 at 5% point	Within	1.0		
		Rows	23.9		
		Columns	2.6		
		Interactions	2.3		
1	0.3	Rows	27.8	28.2	28.0
		Columns	4.6	4.7	4.8
		Interactions		2.3	2.3
2	0.2	Rows	13.1	13.4	13.0
		Columns	13.0	13.3	13.0
		Interactions		5.1	5.1
3	0.2	Rows	32.9	33.3	33.4
		Columns	3.0	3.7	3.5
		Interactions		4.0	4.0
4	0.3	Rows	22.2	21.4	22.2
		Columns	1.2	1.1	1.2
		Interactions		3.7	3.8
5	0.1	Rows	36.2	35.9	36.2
		Columns	0.6	0.4	0.5
		Interactions		2.0	1.9
6	0.1	Rows	18.1	17.9	17.9
		Columns	4.5	4.2	4.3
		Interactions		3.0	3.0
7	0.2	Rows	48.8	47.7	49.2
		Columns	2.9	3.7	3.4
		Interactions		8.9	9.0
8	0.2	Rows	22.0	22.4	22.5
		Columns	4.2	4.7	4.8
		Interactions		3.7	3.7
Average		Rows	27.6	27.5	27.8
		Columns	4.2	4.5	4.4
		Interactions		4.1	4.1

the method of weighted squares of means does not produce the same results as the standard method if subclass numbers are proportional but unequal.

It may be stated without illustration that if the subclass numbers are equal all methods yield the same results, irrespective of whether interactions do or do not exist.

DISCUSSION OF EXPERIMENTAL RESULTS

32. An inspection of tables 19-23 leads to the following conclusions:

A. The differences among the analyses by the several methods are trivial when compared with the differences between the samples and the populations from which they are drawn. From this viewpoint, one method of treatment is practically as good as any other.

B. The variation in possible departure from proportionality and the variation in the values of chi-square seem to have made remarkably little impression on the results. The mean squares actually vary most from theoretical small sample values in tables 19 and 22 where the values of chi-square and the possible departure are least.

C. The mean squares computed by the method of weighted squares of means were almost always less than the corresponding ones by the other methods. They were better estimates of population values in table 19 and as good in tables 20 and 21. Apparently, the method decreased in value somewhat as the disproportion of the subclass numbers increased. But by the same argument, one would be forced to the conclusion that the other methods increased in value with increasing disproportion of the numbers! The experimental results are probably too limited in extent to support either deduction.

TABLE 23. ANALYSES OF SAMPLES DRAWN AT RANDOM FROM A POPULATION DESCRIBED BY A 3×3 TABLE. SERIES II. FIFTY PERCENT POSSIBLE DEPARTURE FROM THEORETICAL SUBCLASS NUMBERS.

Sample	Chi-square	Source of variation	Mean square		
			Original	Expected numbers	Fitting constants
Theoretical	9.5 at 5% point	Within	1.0		
		Rows	23.9		
		Columns	2.6		
		Interactions	2.3		
1	9.5	Rows	19.6	20.3	20.3
		Columns	0.0	0.6	0.7
		Interaction		4.8	4.6
2	4.5	Rows	20.0	20.8	20.9
		Columns	2.1	3.2	3.0
		Interaction		0.8	0.9
3	15.1	Rows	24.6	21.8	23.5
		Columns	2.9	0.9	1.8
		Interaction		5.7	5.3
4	6.3	Rows	22.8	21.5	19.2
		Columns	8.4	5.5	4.7
		Interaction		3.0	2.8
5	2.1	Rows	18.0	18.9	17.4
		Columns	2.0	1.3	1.3
		Interaction		1.9	1.8
6	8.5	Rows	16.3	18.1	16.8
		Columns	2.2	3.6	2.7
		Interaction		1.8	1.6
7	10.0	Rows	30.8	30.1	27.3
		Columns	6.6	2.7	3.1
		Interaction		1.8	1.6
8	5.8	Rows	14.8	13.1	13.3
		Columns	6.1	4.8	4.6
		Interaction		2.5	2.5
Average		Rows	20.9	20.6	19.8
		Columns	3.8	2.8	2.7
		Interaction		2.8	2.6

D. Yates calls attention to the fact that the method of fitting constants affords only a test of significance of interaction, not an estimate. Nevertheless, in these sampling experiments, the mean squares for interactions are just as good estimates of population values as those for the main effects. This is one of the reasons for the statement made in section 21. Another reason appears in the fourth paragraph below.

E. The variability of the experimental results may be assessed in a manner which, while involving some approximations, seems reasonably adequate. The mean squares in the samples may be compared with those in the theoretical small sample by use of the familiar tables of F or z. In doing this, the disproportion in the sample subclass numbers is ignored. Since there is no element of randomness in the theoretical sample (section 28), the corresponding degrees of freedom are taken as infinite. For simplicity, consider only the mean squares *between columns*. That in the theoretical sample of series I is 3.0. Those which are larger in the random samples of this series are compared with 3.0 by use of the ratio,

TABLE 24. DATA CHOSEN ARBITRARILY WITH INTERACTION ZERO AND SUBCLASS NUMBERS DISPROPORTIONATE.

Row	Column						Total	
	A_1		A_2		A_3			
	Number	Mean	Number	Mean	Number	Mean	Number	Mean
B_1	1	2	2	3	3	7	6	4.8
B_2	2	3	4	4	6	8	12	5.8
B_3	3	7	6	8	8	12	17	9.7
Total	6	4.8	12	5.8	17	9.7	35	7.5

$$\frac{\text{mean square in sample}}{3.0} = F = 2.99$$

for the 5 percent point (6, table 35). Hence, any mean square

TABLE 25. ANALYSIS OF VARIANCE OF SAMPLE WITH ZERO INTERACTION AND DISPROPORTIONATE SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Mean squares			
		Original	Expected numbers	Fitting constants	Weighted squares of means
Within subclasses	26	21.9			
Between rows	2	79.3	84.1	84.0	70.0
Between columns	2	79.3	84.1	84.0	70.0
Interactions	4		0.0	0.0	

between columns in the samples of series I is significantly greater than the theoretical value if it is equal to or more than

$$(2.99)\ (3.0) = 9.0.$$

TABLE 26. DATA CHOSEN ARBITRARILY WITH INTERACTION PRESENT AND SUBCLASS NUMBERS PROPORTIONAL.

Row	Column				Tota	
	A_1		A_2			
	Number	Mean	Number	Mean	Number	Mean
B_1	6	8	4	3	10	6
B_2	3	11	2	11	5	11
B_3	3	21	2	11	5	17
Total	12	12	8	7	20	10

A glance through the tables reveals the fact that 2 of the 28 mean squares, or 7.1 percent, exceed this value.

In a similar way, the 5 percent value for mean squares significantly less than 3.0 is computed thus:

$$\frac{3.0}{\text{mean square in sample}} = F = 19.50,$$

$$\text{therefore, mean square in sample} = \frac{3.0}{19.50} = 0.154.$$

Sample 7 in table 19 would have a smaller mean square than this if judged by the method of weighted squares of means. Since the other methods are in the majority, this result will not be counted.

Proceeding in the same way through all the mean squares in all the tables, only 1.9 percent of the mean squares are found to exceed the 5 percent points. This manner of approximation is sufficient to show that the methods of treating disproportionate subclass numbers all give estimates of population values with about the same 5 percent points as would ordinarily be expected in random sampling.

An examination of the illustrative analyses of tables 24 to 27 emphasizes these facts; (i) The fitting of constants leads to perfect estimates of interactions, either zero or greater, in samples where the values of such interactions are deducible from other considerations; and (ii) the estimates of the main effects by the method of weighted squares of means are different from those obtained by the standard method if the subclass numbers are proportional but unequal.

TABLE 27. ANALYSIS OF VARIANCE OF SAMPLE WITH PROPORTIONAL SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Mean squares		
		Standard method and expected numbers	Fitting constants	Weighted squares of means
Between columns	1	120	120	108
Between rows	2	205	205	184
Interaction	2	30	30	30

CONCLUSIONS

A. Very satisfactory amounts of information can be extracted from tables of multiple classification with disproportionate subclass numbers. The results of the experiments described in sections 28-32 varied no more than would be expected in simple sampling with equal or proportional numbers.

B. The various available methods usually yield much the same results. The few necessary precautions are summarized in "Suggestions for using various methods" immediately following.

C. Each method is based on a postulate concerning a population. If it is reasonable to suppose that the sample was derived from a population described by one of these postulates, then the corresponding method of treatment can be used with greater confidence than otherwise.

D. So far as the reported experimental results are representative, the usual methods of testing significance are applicable even when the subclass numbers are disproportionate.

SUGGESTIONS FOR USING VARIOUS METHODS OF ANALYZING THE VARIANCE OF TABLES OF MULTIPLE CLASSIFICATION WITH DISPROPORTIONATE SUBCLASS NUMBERS

I. The method of unweighted means, sections 18-19.
- a. For preliminary surveys, especially if little is known about the population and if the subc'ass numbers do not vary greatly.
- b. For final analysis in cases where the subclass numbers are almost equal.

II. The method of expected subclass numbers, sections 5-9.
- a. If the population subclass numbers are assumed to be proportional or equal.
- b. If there are more than two criteria of classification.

III. The method of weighted squares of means, sections 15-17.
- a. In $2 \times s$ tables in which interactions are assumed to exist.
- b. In other tables of double classification in which no information about interactions is desired.

IV. The method of fitting constants, sections 10-14.
- a. If interactions in the population are assumed to be non-existent, and the most reliable of results is required.
- b. In cases where data are missing from some of the subclasses.
- c. If a test of significance of the interactions is desired where the disproportionate subclass numbers are assumed to be representative of the population. If interaction is non-significant, the main effects are well estimated. See Vb.

V. The ordinary method, appropriate for proportional subclass numbers, section 25.
- a. In rough approximations where information about only the main effects is required.
- b. For examining the main effects where the disproportionate subclass numbers are assumed to be representative of a population in which interactions exist. See IVc.

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(2) The analysis of variance in a 2×s table with disproportionate frequencies. Jour. Am. Stat. Assoc., 28:164. 1933.
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(8) Yates, F. The principles of orthogonality and confounding in replicated experiments. Journal of Agr. Sci., 23:108. 1933.
(9) The analysis of multiple classifications with unequal numbers in the different classes. Jour. Am. Stat. Assoc., 29:51. 1934.

CHRONOLOGICAL ARRANGEMENT OF PUBLICATIONS ON DISPROPORTIONATE SUBCLASS NUMBERS IN TABLES OF MULTIPLE CLASSIFICATION

1931. R. A. Fisher described verbally the application of the method of least squares to the solution of a two-way table. Application of the method had been made by one of his students. No record of publication has been found.

1932. Brown, Bernice. A sampling test of the technique of analyzing variance in a 2×n table with disproportionate frequencies. Proceedings of the Iowa Academy of Science, 39:205.

1932. Brandt, A. E. A statistical study of the relation of sex, breed and live measurements to carcass weights in swine. A thesis submitted for the degree Doctor of Philosophy, Iowa State College.

1932. Brown, Bernice. The evaluation of a statistical technique for analysis of rat feeding data, based on uniformity trials. A thesis submitted for the degree Master of Science, Iowa State College.

1933. Yates, F. The principles of orthogonality and confounding in replicated experiments. Jour. Agr. Sci., 23:108.

1933. Brandt, A. E. The analysis of variance in a 2×s table with disproportionate frequencies. Jour. Am. Stat. Assoc., 28:164.

1933. Becker, E. R., and Hall, Phoebe, R. The possible role of inheritance in the quantitative character of a coccidian infection of the rat. Parasitology, 25:397.

1934. Snedecor, George W. Calculation and interpretation of analysis of variance and covariance. Collegiate Press, Inc., Ames, Iowa.

1934. Yates, F. The analysis of multiple classification with unequal numbers in the different classes. Jour. Am. Stat. Assoc., 29:51.

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1934. Hendricks, Walter A. Analysis of variance considered as an application of simple error theory. Accepted for publication in The Annals of Mathematical Statistics, and expected to appear in the issue of December, 1934.

TABLE 28. NUMBER OF HENS AND MEAN WEIGHT (GRAMS) OF EGGS PRODUCED. EACH WEIGHT DECREASED BY 47 GRAMS.

Month	Year		Pen number								Total Expected
			1		2		3		4		
			Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected	
January	30	Number	23	20.23	25	22.72	26	22.72	22	21.08	86.74
		Weight	1.85	37.47	2.52	57.33	3.05	69.37	.55	11.59	175.76
	31	Number	21	23.74	25	26.66	22	26.66	22	24.74	101.80
		Weight	8.87	210.49	9.34	248.89	8.58	228.66	8.99	222.31	910.35
	32	Number	24	21.56	22	24.22	25	24.22	24	22.47	92.46
		Weight	9.86	212.58	8.74	211.77	7.20	174.45	7.28	163.66	762.46
	Total	Number	68	65.53	72	73.59	73	73.59	68	68.29	281
		Weight		460.54		518.00		472.48		397.56	1,848.58
March	30	Number	19	20.95	23	23.52	26	23.52	19	21.83	89.82
		Weight	3.09	64.72	4.30	101.05	5.26	123.68	3.60	78.70	368.15
	31	Number	26	24.58	31	27.61	27	27.61	28	25.62	105.42
		Weight	8.45	207.74	8.52	235.11	7.37	203.58	7.04	180.34	826.77
	32	Number	23	22.33	25	25.08	23	25.08	21	23.27	95.76
		Weight	11.28	251.95	9.16	229.81	7.54	189.17	7.68	178.74	849.67
	Total	Number	68	67.86	79	76.21	76	76.21	68	70.72	291
		Weight		524.41		565.97		516.44		437.78	2,044.59
May	30	Number	19	20.16	23	22.63	24	22.63	17	21.00	86.43
		Weight	3.73	75.21	3.54	80.11	5.90	133.45	3.87	81.30	370.07
	31	Number	25	23.66	29	26.56	24	26.56	27	24.65	101.43
		Weight	7.13	168.61	6.49	172.39	5.72	151.86	5.70	140.42	633.28
	32	Number	21	21.49	22	24.13	25	24.13	24	22.39	92.14
		Weight	9.46	203.20	6.52	157.28	6.41	154.72	6.33	141.81	657.00
	Total	Number	65	65.30	74	73.33	73	73.33	68	68.04	280
		Weight		447.03		409.78		440.03		363.52	1,660.36

Month	Year		Pen number 1 Actual	Pen number 1 Expected	Pen number 2 Actual	Pen number 2 Expected	Pen number 3 Actual	Pen number 3 Expected	Pen number 4 Actual	Pen number 4 Expected	Total Expected
July	30	Number	17	19.44	22	21.83	24	21.83	17	20.25	83.34
		Weight	2.07	40.24	2.06	44.94	1.68	36.74	1.89	38.24	160.17
	31	Number	22	22.81	26	25.62	24	25.62	28	23.77	97.81
		Weight	5.33	121.62	5.68	145.61	4.61	118.15	4.16	98.90	484.28
	32	Number	20	20.72	23	23.27	24	23.27	23	21.59	88.84
		Weight	6.88	142.65	4.00	93.17	4.41	102.57	2.79	60.17	398.57
	Total	Number	59	62.97	71	70.71	72	70.71	68	65.62	270
		Weight		304.52		283.73		257.46		197.32	1,043.02
September	30	Number	14	14.61	17	16.41	17	16.41	15	15.23	62.66
		Weight	3.60	52.61	5.18	85.04	6.22	102.03	5.19	78.98	318.67
	31	Number	20	17.15	18	19.26	18	19.26	17	17.87	73.54
		Weight	6.58	112.76	7.10	136.85	4.84	93.30	5.76	103.02	445.93
	32	Number	15	15.58	16	17.49	18	17.49	18	16.23	66.80
		Weight	7.97	124.10	6.14	107.48	6.20	108.46	5.55	90.09	430.14
	Total	Number	49	47.34	51	53.16	53	53.16	50	49.33	203
		Weight		289.47		329.37		303.79		272.10	1,194.74
Year	30	Number	92	95.38	110	107.11	117	107.11	90	99.39	409
		Weight		270:25		368.48		465.28		288.82	1,392.83
	31	Number	114	111.94	129	125.70	115	125.70	122	116.65	480
		Weight		821.23		938.85		795.56		744.99	3,300.62
	32	Number	103	101.68	108	114.18	115	114.18	110	105.96	436
		Weight		934.49		799.50		729.36		634.48	3,097.84
	Total	Number	309		347		347		322		1,325
		Weight		2,025.97		2,106.84		1,990.19		1,668.28	7,791.29

APPENDIX I

THE METHOD OF EXPECTED SUBCLASS NUMBERS APPLIED TO A TABLE WITH THREE CRITERIA OF CLASSIFICATION

The computation of proportional numbers in a table with multiple classification involves no new theory, but is increasingly intricate as the number of criteria of classification increases. For illustration, the method is applied to the data in table 28. These data were extracted from more extensive unpublished tables made available by E. W. Henderson, head of the Poultry Husbandry Subsection of the Iowa Agricultural Experiment Station. The actual numbers are entered on the left side of the double column under each pen number. For convenience in calculation, each mean is decreased by 47 grams.

Roughly, the numbers of hens are characteristic of the months, pens and years. The hypothesis of proportional subclass numbers seems reasonable. The value of chi-square is obviously small for the number of subclasses.

The first expected numbers to be calculated are those for the 3-year totals by pen and month. The method is the same as that of section 7. In pen 1, for example, the ratio of the pen total to the total for all pens, 309/1325, is multiplied successively by the month totals, 281, 291, etc. Thus, the expected number for January is

$$(309/1325)281 = 65.53.$$

This is done for each of the 20 pen-month subclasses. Verification consists in summing the expected numbers in the pens and in the months, the sums being equal to the actual class numbers.

For present purposes, it is sufficient to report the numbers in table 28 with four significant figures. The computations were all made with six or more figures. While no difficulty in following the explanations will be experienced, anyone who duplicates the calculations will have to carry his numbers more exactly in order to verify the results given.

Next are calculated the expected numbers for the individual years. This is done by dividing each pen-month total into three parts proportional to the

TABLE 29. EXPECTED NUMBERS OF HENS AND SUMS OF MEAN EGG WEIGHTS. PEN-YEAR.

Pen	Year						Total	
	1930		1931		1932			
	Number	Sum	Number	Sum	Number	Sum	Number	Sum
1	95.38	270.25	111.94	821.23	101.68	934.49	309.00	2025.97
2	107.11	368.48	125.71	938.85	114.18	799.51	347.00	2106.84
3	107.11	465.28	125.71	795.55	114.18	729.36	347.00	1990.19
4	99.40	288.82	116.64	744.99	105.96	634.48	322.00	1668.28
Total	409.00	1392.83	480.00	3300.62	436.00	3097.84	1325.00	7791.29

COMPUTATIONS

Correction term: $(7791.29)^2/1325 = 45814.44.$

Between subclasses:
$$(270.25)^2/95.38 + \quad . \quad . \quad . \quad + (634.48)^2/105.96 - 45814.44 = 4555.70.$$

Between means of pens:
$$(2025.97)^2/309 + \quad . \quad . \quad . \quad + (1668.28)^2/322 - 45814.44 = 318.74.$$

Between means of years:
$$(1392.83)^2/409 + \quad . \quad . \quad . \quad + (3097.84)^2/436 - 45814.44 = 3635.32.$$

Pen-year interactions:
$$4555.70 - (318.74 + 3635.32) = 601.64.$$

TABLE 30. ANALYSIS OF VARIANCE OF MEAN EGG WEIGHTS. METHOD OF EXPECTED SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Between means of pens	3	318.7	106
Between means of months	4	1,618.4	405
Between means of years	2	3,635.3	1,818
First order interactions:			
Pen-month	12	66.6	6
Pen-year	6	601.6	100
Month-year	8	1,060.1	133
Second order interactions	24	171.8	7
Total between 60 subclasses	59	7,472.5	
Within subclasses	1,265		14
Total	1,324		

annual totals. Using for illustration the January expected total number for pen 1, 65.53, multiply successively by 409/1325, 480/1325 and 436/1325, the resulting expected subclass numbers being 20.23, 23.74 and 21.56. The same three multipliers are used in all pen-month groups.

The expected sums are now calculated by multiplying each expected number by the corresponding mean. In pen 1 for January, 1930,

$$(20.23)(1.85) = 37.47,$$

and so on.

The expected numbers and sums are now combined in the usual manner to complete the analysis of variance. (The method is explained in examples 6 and 9 of reference 6.) The sum of squares between 60 subclasses is first computed. Then three tables like table 29 are compiled. The other two are the pen-month and year-month summaries. From these tables the main effects and first order interactions are computed. The sum of squares for the second order interactions is the remainder in table 30, determined by subtracting from the *total between* sum of squares all those for the main effects and first order interactions. The sum of squares *within subclasses* was derived from the original mean egg weights of the 1325 hens.

APPENDIX II

FITTING CONSTANTS IN A 5×9 TABLE

This example illustrates the general method of fitting constants in a two-way table. The data in table 31, collected by the Division of Crop and Livestock Estimates, Bureau of Agricultural Economics, U.S.D.A., were made available through the courtesy of L. M. Carl, Agricultural Statistician for Iowa.

From the statistical viewpoint there are two notable features of the table—the subclass numbers are not proportional, and the only special hypothesis about the population which seems reasonable is that the interaction may be negligible. The postulates of proportional and equal subclass numbers are untenable. The actual numbers are characteristic of the districts. In some, the number of farms reaches a maximum in the 60-89 acre size; in others, the majority of the farms are small. The inappropriateness of the method of expected numbers is indicated by the fact that chi-square for this table is 348.

The ratios are computed and verified as in section 11. In a table of original entry, the subclass sums would probably appear, but they have been omitted from table 31 in the interest of clarity. The only sums and means necessary for the present computation are those in the borders.

TABLE 31. NUMBER OF FARMS AND YIELD OF CORN ON FARMS OF DIFFERENT SIZE IN NINE DISTRICTS OF IOWA, 1933. EACH YIELD IS DECREASED BY 30 BUSHELS PER ACRE.

District constants		Size of farms (acres) and size constants					Number	Yields		b + m
		0–29 a_1	30–59 a_2	60–89 a_3	90–149 a_4	150 up a_5		Sum	Mean	
b_1	Number	7	68	71	55	16	217	2,928	13.4931	13.3508
	Ratio	0.032258	0.313364	0.327189	0.253456	0.073733				
b_2	Number	32	86	68	44	9	239	3,980	16.6527	16.6751
	Ratio	0.133891	0.359833	0.284519	0.184100	0.037657				
b_3	Number	100	97	31	9	2	239	2,741	11.4686	11.8721
	Ratio	0.418410	0.405858	0.129707	0.037657	0.008368				
b_4	Number	18	71	98	67	28	282	3,044	10.7943	10.6836
	Ratio	0.063830	0.251773	0.347518	0.237588	0.099291				
b_5	Number	35	104	92	48	22	301	5,428	18.0332	18.0642
	Ratio	0.116279	0.345515	0.305648	0.159468	0.073090				
b_6	Number	57	115	48	32	7	259	4,914	18.9730	19.1342
	Ratio	0.220077	0.444016	0.185328	0.123552	0.027027				
b_7	Number	24	73	58	49	24	228	3,578	15.6930	15.6397
	Ratio	0.105263	0.320176	0.254386	0.214912	0.105263				
b_8	Number	73	81	43	22	9	228	1,112	4.8772	5.1322
	Ratio	0.320176	0.355263	0.188596	0.096491	0.039474				
b_9	Number	59	80	51	19	2	211	2,118	10.0379	10.2818
	Ratio	0.279621	0.379147	0.241706	0.090047	0.009479				
	Number	405	775	560	345	119	2,204			
	Sum	4751	10560	7546	5214	1672		29,843		
	Mean	11.7309	13.6258	13.6536	15.1130	14.0504				13.5404
	a	−0.82740	−0.19498	−0.14884	1.07066	0.10056				

TABLE 32. COEFFICIENTS OF THE a-EQUATIONS.

Equation	a_1	a_2	a_3	a_4	a_5	Constant
a_1	−298.489	152.216	86.016	45.846	14.410	255.023
a_2	152.216	−496.025	190.256	115.091	38.463	69.544
a_3	86.016	190.256	−407.138	96.773	34.093	59.371
a_4	45.846	115.091	96.773	−280.730	23.020	−373.028
a_5	14.410	38.463	34.093	23.020	−109.987	− 10.911
Total	− 0.001	0.001	0.000	− 0.000	− 0.001	− 0.001

The five a-equations are written as follows:

$$
\begin{aligned}
a_1:\;& [(7)\ (0.032258) + (32)\ (0.133891) +\ldots\ (59)\ (0.279621) - 405]a_1 \\
&+ [(7)\ (0.313364) + (32)\ (0.359833) +\ldots\ (59)\ (0.379147)]a_2 \\
&+ \ldots \qquad \ldots \qquad \ldots \\
&+ [(7)\ (0.073733) + (32)\ (0.037657) +\ldots\ (59)\ (0.009479)]a_5 \\
&= (7)\ (13.4931) + (32)\ (16.6527) +\ldots\ (59)\ (10.0379) - 4751.
\end{aligned}
$$

$$
\begin{aligned}
a_2:\;& [(68)\ (0.032258) + (86)\ (0.133891) +\ldots + (80)\ (0.279621)]a_1 \\
&+ [(68)\ (0.313364) + (86)\ (0.359833) +\ldots + (80)\ (0.379147) - 775]a_2 \\
&+ \ldots \qquad \ldots \qquad \ldots \quad \ldots \\
&+ [(68)\ (0.073733) + (86)\ (0.037657) +\ldots + (80)\ (0.009479)]a_5 \\
&= (68)\ (13.4931) + (86)\ (16.6527) +\ldots + (80)\ (10.0379) - 10560.
\end{aligned}
$$

etc.

$$
\begin{aligned}
a_5:\;& [(16)\ (0.032258) + (9)\ (0.133891) +\ldots + (2)\ (0.279621)]a_1 \\
&+ \ldots \qquad \ldots \qquad \ldots \quad \ldots \\
&+ [(16)\ (0.073733) + (9)\ (0.037657) +\ldots + (2)\ (0.009479) - 119]a_5 \\
&= (16)\ (13.4931) + (9)\ (16.6527) +\ldots + (2)\ (10.0379) - 1672.
\end{aligned}
$$

TABLE 33. COEFFICIENTS OF a-EQUATIONS AFTER a_5 IS ELIMINATED.

Equation	a_1	a_2	a_3	a_4	Constant
a_1	−312.899	137.806	71.605	31.436	255.023
a_2	113.753	−534.488	151.793	76.628	69.544
a_3	51.923	156.163	−441.231	62.680	59.371
a_4	22.827	92.071	73.754	−303.749	−373.028

In the formation of these equations, the following points may be noted:

1. All five of the a's are present in each equation.
2. In the a_1-equation, the subclass numbers in the a_1 farm size appear again and again. In the a_2-equation, the numbers are those in the second farm size, etc.
3. In the a_1-equation, the first class number, 405, is subtracted in the coefficient of a_1. In the next equation, the second class number, 775, is subtracted in the coefficient of a_2, etc.
4. The coefficients in the equations, excepting those along the diagonal, occur in identical pairs (see table 32). For example, in the a_1-equation the coeffi-

TABLE 34. COEFFICIENTS OF EQUATIONS AFTER DIVISION BY LEADING COEFFICIENT.

Equation	a_1	a_2	a_3	a_4	Constant
a_1	1	−0.440415	−0.228845	−0.100466	−0.815032
a_2	1	−4.698679	1.334410	0.673632	0.611365
a_3	1	3.007607	−8.497867	1.207184	1.143450
a_4	1	4.033504	3.231050	−13.306826	−16.341814

TABLE 35. COEFFICIENTS OF EQUATIONS AFTER a_1 IS ELIMINATED.

Sum equation	a_2	a_3	a_4	Constant
1	4.258264	−1.563255	−0.774098	−1.426397
2	−3.448022	8.269022	−1.307650	−1.958482
3	−4.473919	−3.459895	13.206360	15.526782

cient of a_2 is the same as that of a_1 in the a_2-equation. A check on accuracy is provided by calculating all these coefficients independently. Other checks are to be indicated later, however, so the additional labor is usually unnecessary.

As the successive terms in the equations are computed, they are entered in table 32. If the preceding calculations are accurate, the sum of the coefficients in each column of the table is zero. Owing to the rounding of the numbers in the last place, this sum is seldom exact.

Since the sum of the a-constants is zero, the first step in the solution of these equations, is to make the substitution.

$$a_5 = -(a_1 + a_2 + a_3 + a_4).$$

This is done by subtracting the coefficient of a_5 in each equation from each of

TABLE 36. STEPS IN THE ELIMINATION OF a_2.

1. Divide by coefficients of a_2.

Equation	a_2	a_3	a_4	Constant
1	1	−0.367111	−0.181787	−0.334971
2	1	−2.398193	0.379246	0.568002
3	1	0.773348	−2.951855	−3.470510
2. Subtract equations 2 and 3 from 1.				
1		2.031082	−0.561033	−0.902973
2		−1.140459	2.770068	3.135539

the other coefficients (but not the constant) in the same equation. For example, in the a_1-equation,

$$-298.489 - 14.410 = -312.899,\ 152.216 - 14.410 = 137.806, \text{ etc.}$$

The results are recorded in table 33. Equation a_5 (or any one of the others) is superfluous, and is dropped.

The next step toward the solution of the equations is the division of each by its leading coefficient. As an example, each term of the first equation is divided by -312.899. The results appear in table 34.

Next comes the elimination of the a_1's from the equations. This is done by subtracting each of the last three equations in succession from the a_1-equation. This reduces the coefficient of a_1 in each difference to zero.

The a_2-coefficient in the first difference is,

$$-0.440415 - (-4.698679) = 4.258264,$$

TABLE 37. STEPS IN THE ELIMINATION OF a_3.

1. Divide by coefficients of a_3.

Equation	a_3	a_4	Constant
1	1	−0.276224	−0.444577
2	1	−2.428906	−2.749366
2. Subtract equation 2 from 1.			
		2.152682	2.304789

while the a_3-coefficient is,

$$-0.228845 - 1.334410 = -1.563255$$

In the second difference, the coefficient of a_2 is,

$$-0.440415 - 3.007607 = -3.448022.$$

The results are entered in table 35.

After this point is reached, the foregoing process is repeated until the value of the last a is determined. Table 36 constitutes the second cycle, resulting in the elimination of a_2. In the same way, a_3 is eliminated in table 37, leaving the single equation,

$$2.152682a_4 = 2.304789,$$

from which,

$$a_4 = 2.304789/2.152682 = 1.07066.$$

TABLE 38. PARTIALLY COMPUTED ANALYSIS OF VARIANCE OF CORN YIELDS, USING DISPROPORTIONATE SUBCLASS NUMBERS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Within subclasses	2,159	302,985	140.34
Between means of districts	8	39,942	4,992.74
Between means of farm sizes	4	2,222	555.42
Total between subclasses	44	45,860	1,042.28

This value is now entered in the proper place in the last line of table 31. Also, a_4 substituted in one of the equations in the first part of table 37, allows the solution for a_3. For example, using the second equation, multiply the coefficient of a_4, -2.428906, by 1.07066, change the sign of the product, and add it to the constant term, -2.749366. That is,

$$a_3 = (2.428906)(1.07066) - 2.749366 = -0.14884.$$

Returning now to one of the equations in the first part of table 36, the third for example, the coefficients of a_3 and a_4 are multiplied by the respective values just computed, the signs changed, and the products added to the constant term to yield the value of a_2. That is,

$$a_2 = (-0.773348)(-0.14884) + (2.951855)(1.07066) - 3.470510 = -0.19498.$$

In a similar manner, using an equation of table 34, say the fourth,

$$a_1 = (-4.033504)(-0.19498) + (-3.231050)(-0.14884) + (13.306826)(1.07066) - 16.341814 = -0.82740.$$

TABLE 39. ANALYSIS OF VARIANCE OF CORN YIELDS. METHOD OF FITTING CONSTANTS.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Within subclasses	2,159		140
Between means of districts	8	38,354	4,794
Between means of farm sizes	4	634	158
Interactions	32	5,284	165

The value of a_5 is now computed by the relation,

$$a_5 = -(-0.82740 - 0.19498 - 0.14884 + 1.07066) = 0.10056.$$

It remains to verify the results so far computed by substituting all the a's in the equations of table 32. Using the a_1 equation, for example,

$$(-298.489)(-0.82740) + (152.216)(-0.19498) + (86.016)(-0.14884) + (45.846)(1.07066) + (14.410)(0.10056) = 255.023.$$

The results should check approximately to four significant figures. It is better to verify the results in all the equations, though usually one is enough.

Returning now to table 31, using the means of the rows, the ratios and the a's, the values of b +m are computed thus:—

$$b_1 + m = 13.4931 - [(0.032258)(-0.82740) + (0.313364)(-0.19498) + (0.327189)(-0.14884) + (0.253456)(1.07066) + (0.073733)(0.10056)] = 13.35080$$

$$b_2 + m = 16.6527 - [(0.133891)(-0.82740) + (0.359833)(-0.19498) + (0.284519)(-0.14884) + (0.184100)(1.07066) + (0.037657)(0.10056)] = 16.67509$$

Similarly,

$$\begin{aligned} b_3 + m &= 11.87208 \\ b_4 + m &= 10.68356 \\ b_5 + m &= 18.06418 \\ b_6 + m &= 19.13425 \\ b_7 + m &= 15.63970 \\ b_8 + m &= 5.13218 \\ b_9 + m &= 10.28180 \end{aligned}$$

These results are entered in the righthand column of table 31. The correction term is computed,

$$(29843)^2/2204 = 404{,}086$$

The *reduction in sum of squares due to fitting constants* is calculated by going across the bottom rows and down the righthand columns of the table 31, multiplying the constants by the corresponding sums, and deducting the correction term:

$$(-0.82740)(4751) + (-0.19498)(10560) + \ldots + (0.10056)(1672) + (13.35080)(2928) + \ldots + (10.28180)(2118) - 404086 = 40576.$$

The partially computed analysis of variance, using the individual farm yields of corn, was computed in the manner of example 6, reference 6. The results are shown in table 38. The last item in this table less the *reduction in sum of squares* computed above (section 12) yields the sum of squares for testing interactions:

$$45860 - 40576 = 5284.$$

The correction for disproportionate subclass numbers (section 12) is

$$39942 + 2222 - 40576 = 1588.$$

Subtracting this from the sums of squares for the main effects in table 38, the appropriate quantities for table 39 are calculated.

Owing to the inappropriateness of postulating a population with proportional subclass numbers in this example the method of expected numbers, applied to these corn yield data, gives results significantly different from those in table 39. It is the only case which has been found in which this is true.

The hypothesis of nonexistent interaction appears to be justified. However, another question has been raised. If no hypothesis is made concerning population, the mean square *between farm sizes* is significantly greater than experimental error, since $F = 555/140 = 3.96$. But under the hypothesis of nonexistent interaction, the corresponding mean square is not significant. While the evidence is in favor of the hypothesis, the question cannot be finally answered on the basis of the present analysis.